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THE NUMBER OF TRIANGLES IN A K_4-FREE GRAPH

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We show that a K_4-free graph with e edges has at most $(e/3)^{\frac{3}{2}}$ triangles. This is strict for the disjoint union of a complete balanced tripartite graph with any number of isolated nodes. In Fig 1, this bound is compared with other results.

Theorem 1. *If a K_4-free graph has e edges and t triangles, then $t \leqslant (e/3)^{\frac{3}{2}}$.*

Proof. Let V, E and T be the nodes, edges and triangles of a K_4-free graph, G. For each $a \in V$, let N_a be the neighbors of a and let P_a be the edges with both ends in N_a. Since each edge (a, b) is in $|N_a \cap N_b|$ triangles, $t = \frac{1}{3} \sum_{(a,b)\in E} |N_a \cap N_b|$. Using the Cauchy–Schwarz inequality,

$$t^2 = \frac{1}{9}\Big(\sum_{(a,b)\in E} |N_a \cap N_b|\Big)^2 \leqslant \frac{e}{9} \sum_{(a,b)\in E} |N_a \cap N_b|^2.$$

Again since edge (a, b) is in $|N_a \cap N_b|$ triangles and using the fact that

$$2(|A \cap B| + |A \cap C| + |B \cap C|) \leqslant |A| + |B| + |C| + 3\,|A \cap B \cap C|$$

gives

$$t^2 \leqslant \frac{e}{9} \sum_{(a,b)\in E} |N_a \cap N_b|^2 = \frac{e}{9} \sum_{(a,b,c)\in T} (|N_a \cap N_b| + |N_a \cap N_c| + |N_b \cap N_c|)$$
$$\leqslant \frac{e}{18} \sum_{(a,b,c)\in T} (|N_a| + |N_b| + |N_c| + 3\,|N_a \cap N_b \cap N_c|).$$

Since G is K_4-free, $N_a \cap N_b \cap N_c = \emptyset$ for $(a, b, c) \in T$. Because node a is in $|P_a|$ triangles and $|N_a|$ edges,

$$t^2 \leqslant \frac{e}{18} \sum_{(a,b,c)\in T} (|N_a| + |N_b| + |N_c|) = \frac{e}{18} \sum_{a\in V} |P_a|\,|N_a| = \frac{e}{18} \sum_{(a,b)\in E} (|P_a| + |P_b|).$$

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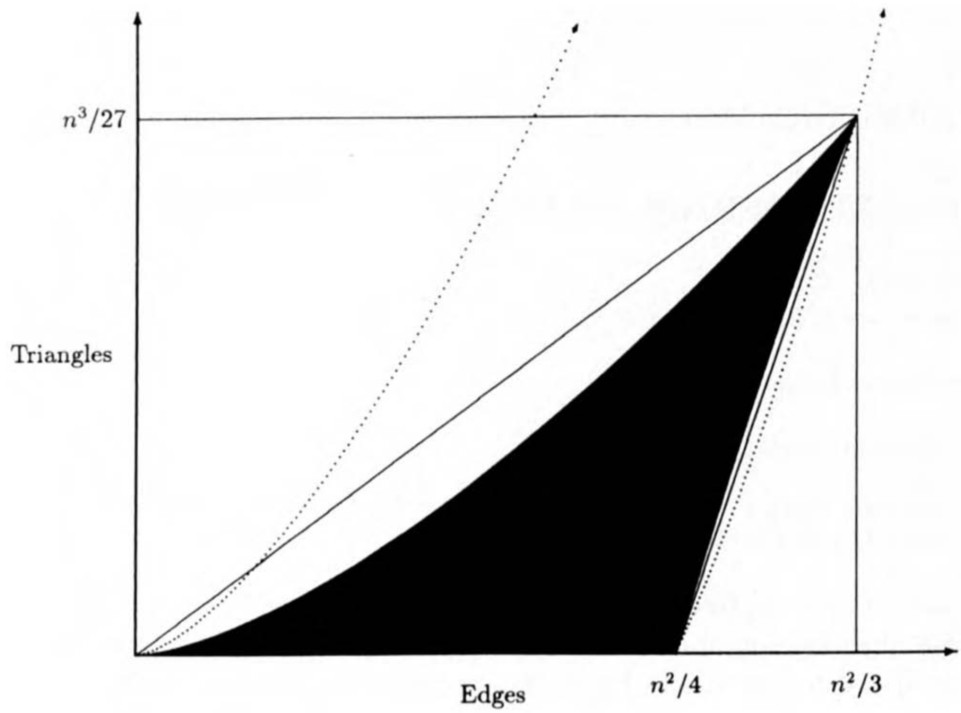

Fig. 1. Feasible K_4-free graphs: Let n, e, and t be the number of nodes, edges and triangles in a graph. The dark area shows the possible combinations of these in a K_4-free graph. It is bounded on the left by $t \leqslant (e/3)^{\frac{3}{2}}$, on the right by $t \geqslant (9en - 2n^3 - 2(n^2 - 3e)^{\frac{3}{2}})/27$ (Fisher and Solow [3]), and below by the trivial $t \geqslant 0$. These are strict (up to lower order terms) and imply $t \leqslant n^3/27$ (horizontal line – Moon and Moser [5]), $e \leqslant n^2/3$ (vertical line – Turán [8]), $t \leqslant (2e)^{\frac{3}{2}}/6$ (left curve – Erdös and Hanini [2]), $t \geqslant (4e - n^2)e/(3n)$ (right curve – Nordhaus and Stewart [7]), $t \leqslant en/9$ (left diagonal line – Moon and Moser [5]), and $t \geqslant (4e - n^2)n/9$ (right diagonal line – Bollobás [1]). The curves and the right diagonal line are also bounds for all (not necessarily K_4-free) graphs. While the left curve is strict (up to lower order terms), no known graph lies to the right of the dark area.

One other important bound is $t \geqslant (e - \lfloor n^2/4 \rfloor)\lfloor n/2 \rfloor$ for $n^2/4 \leqslant e < \lfloor n^2/4 \rfloor + \lfloor n/2 \rfloor$ (Nikiforov and Khadzhiivanov [6] and Lovász and Simonovits [4]). This would appear as a vanishingly short (as $n \to \infty$) line tangent to the dark area's right boundary at $e = n^2/4$.

Again since G is K_4-free, $P_a \cap P_b = \emptyset$ for $(a, b) \in E$. So if $(a, b, c) \in T$, then $e \geqslant |P_a \cup P_b \cup P_c| = |P_a| + |P_b| + |P_c|$. Pick $(r, s) \in E$ so $u \equiv |P_r| + |P_s| \geqslant |P_a| + |P_b|$ for all $(a, b) \in E$. Thus for $(a, b) \in P_r$, $(a, b) \in P_s$ and $(a, b) \in E - P_r - P_s$, we have $|P_a| + |P_b| \leqslant e - |P_r|$, $|P_a| + |P_b| \leqslant e - |P_s|$ and $|P_a| + |P_b| \leqslant u$, respectively. So,

$$t^2 \leqslant \frac{e}{18}\Bigg(\sum_{(a,b)\in P_r}(|P_a| + |P_b|) + \sum_{(a,b)\in P_s}(|P_a| + |P_b|) + \sum_{(a,b)\in E-P_r-P_s}(|P_a| + |P_b|)\Bigg)$$

$$\leqslant \frac{e}{18}(|P_r|(e - |P_r|) + |P_s|(e - |P_s|) + (e - u)u) \leqslant \frac{e}{18}(2ue - 3u^2/2).$$

This is maximized when $u = 2e/3$, giving $t^2 \leqslant e^3/27$. □

Theorem 1 is not strict, e.g., if $e = 9$, Theorem 1 gives $t \leqslant 5$ while all K_4-free graphs with 9 edges have at most 4 triangles. However, Theorem 1 is asymptotically strict for the following family of graphs. For each e, let G_e have nodes labelled $0, 1, \ldots, \lfloor\sqrt{3e}\rfloor$ (where $\lfloor x\rfloor$ is the largest integer less than or equal to x). For $i \neq 0 \neq j$, connect node i to node j if and only if $i-j$ is not a multiple of 3. Connect node 0 to the first k nodes that are not multiples of 3 so that the total number of edges is e. Since G_e is tripartite, it is K_4-free. Also, it can be shown that G_e has

$$\left\lfloor \frac{e^{\frac{3}{2}}}{3\sqrt{3}} - \frac{(1-\beta)\beta}{3} e + \frac{(1-\beta)(\beta - 3\alpha + \beta^2 - 2\alpha\beta + 3\alpha^2)}{2\sqrt{3}} \sqrt{e} + \frac{1}{2} \right\rfloor$$

triangles where $\alpha \equiv \sqrt{e/3} - \lfloor\sqrt{e/3}\rfloor$ and $\beta \equiv \sqrt{3e} - \lfloor\sqrt{3e}\rfloor$.

References

[1] B. Bollobás, Extremal Graph Theory (Academic Press, New York, 1978).

[2] P. Erdös, On the number of complete subgraphs contained in certain graphs, Publ. Math. Inst. Hungarian Acad. Sci. 7 (1962) 459–464.

[3] D.C. Fisher and A.E. Solow, Clique polynomials, Tech. Rep. 87-8, Department of Mathematics, Harvey Mudd College, Claremont, CA (1987).

[4] L. Lovász and M. Simonovits, On complete subgraphs of a graph II, Stud. Pure Math. (1983) 459–496.

[5] J.W. Moon and L. Moser, On a problem of Turán, Publ. Math. Inst. Hungarian Acad. Sci. 7 (1962) 283–286.

[6] V.S. Nikiforov and N.G. Khadzhiivanov, Solution of the problem of P. Erdös on the number of triangles in graphs with n vertices and $[n^2/4]+l$ edges, C.R. Acad. Bulgare Sci. 34, No. 7 (1981) 969–970.

[7] E.A. Nordhaus, and B.M. Stewart, Triangles in an ordinary graph, Canad. J. Math. 15 (1963) 33–41.

[8] P. Turán, On the theory of graphs, Colloq. Math. 3 (1954) 19–30.